Int J Theor Phys (2008) 47: 553–560
DOI 10.1007/s10773-007-9480-5

N-Solitonic Solution in Terms of Wronskian Determinant for a Perturbed Variable-Coefficient Korteweg-de Vries Equation

Cheng Zhang · Hong-Wu Zhu · Chun-Yi Zhang · Zhen-Zhi Yao · Xing Lü · Xiang-Hua Meng · Bo Tian

Received: 20 April 2007 / Accepted: 29 June 2007 / Published online: 24 July 2007

Abstract With the help of symbolic computation, we first derive the bilinear form and an auto-Bäcklund transformation for a perturbed variable-coefficient Korteweg-de Vries equation in this paper. We also construct the N-solitonic solution in the Wronskian form and give the corresponding proof via the Wronskian technique. Furthermore, the authors verify that the $(N-1)$- and N-solitonic solutions indeed satisfy the auto-Bäcklund transformation.

Keywords Variable-coefficient Korteweg-de Vries equation · Auto-Bäcklund transformation · N-solitonic solution · Wronskian determinant

1 Introduction

In recent years, there has been a growing interest in studying variable-coefficient Korteweg-de Vries (vcKdV) models with different forms, and some exact analytic solutions and integrable properties have been established in [1–9]. Considering that coefficient functions are able to reflect the slowly-varying inhomogeneities and nonuniformities of boundaries, the vcKdV models can be used in many important physical and dynamic processes such as the propagation through fluid-filled tubes with elastic walls and tapering effects [10, 11],

C. Zhang · H.-W. Zhu · Z.-Z. Yao · X. Lü · X.-H. Meng · B. Tian (✉)
School of Science, Beijing University of Posts and Telecommunications, P.O. Box 122, Beijing 100876, China
e-mail: gaoyt@public.bta.net.cn

B. Tian
Key Laboratory of Optical Communication and Lightwave Technologies, Ministry of Education, Beijing University of Posts and Telecommunications, Beijing 100876, China

C.-Y. Zhang
Ministry-of-Education Key Laboratory of Fluid Mechanics and National Laboratory for Computational Fluid Dynamics, Beijing University of Aeronautics and Astronautics, Beijing 100083, China

C.-Y. Zhang
Meteorology Center of Air Force Command Post, Changchun 130051, China

motion of pressure pulses in fluid-filled distensible tubes [12–17], Bose–Einstein condensates in the weakly-interacting atomic gases [18–21], cylindrical dust-acoustic and dust-ion-acoustic waves in an un-magnetized dusty plasma [22–25].

As one of the most important vcKdV models, the perturbed vcKdV equation [26] is of the form:

$$u_t + f(t)uu_x + g(t)u_{xxx} + l(t)u = 0, \tag{1}$$

where the wave amplitude $u(x,t)$ is a function of the scaled "space" x and "time" t, $f(t) \neq 0$, $g(t) \neq 0$ and $l(t)$ represent the coefficients of the nonlinear, dispersive and perturbed terms, respectively. Applications of (1) include the nonlinear excitations of a Bose gas of impenetrable bosons with longitudinal confinement [11, 26], dynamics of a circular rod composed of a general compressible hyperelastic material with variable cross-sections and material density [27], propagation of weakly nonlinear solitonic waves in a varied-depth shallow-water tunnel [28, 29], and evolution of internal gravity waves in lakes of changing cross-section [30].

As we know, the Hirota's direct method [31] and Wronskian technique [32, 33] are two important tools to deal with nonlinear evolution equations (NLEEs) and soliton problems: the former can be used to effectively construct the N-soliton solution in the form of Nth-order polynomial in N exponentials for a large class of NLEEs; while the latter provides a simple and straightforward way of verifying the validity of the N-soliton solution by virtue of properties of the Wronskian determinant.

With the help of symbolic computation [34–47], the structure of this paper is organized as follows: Sect. 2 gives the bilinear form and an auto-Bäcklund transformation (BT) for (1). In Sect. 3, we construct the N-solitonic solution in the Wronskian form and prove its validity via the Wronskian technique. Furthermore, Sect. 4 verifies that the auto-BT between the $(N-1)$- and N-solitonic solutions is indeed satisfied. Finally, Sect. 5 offers the conclusions.

2 Bilinear Form and an Auto-BT for (1)

By introducing a potential function $w(x,t)$ for $u(x,t)$, $u = w_x$, it follows from (1) that w satisfies the equation

$$w_t + \frac{1}{2} f(t) w_x^2 + g(t) w_{xxx} + l(t) w = 0. \tag{2}$$

Through the dependent variable transformation

$$w = x^2 A(t) + 12a(t) \frac{\partial}{\partial x} \ln[F(x,t)], \tag{3}$$

(2) becomes

$$\begin{aligned} &\frac{x^2}{6a(t)}[F^2 l(t)A(t) + F^2 A'(t) + 2f(t)F^2 A^2(t)] + \left[2l(t) + \frac{2a'(t)}{a(t)}\right] F F_x \\ &\quad + 2FF_{xt} - 2F_x F_t + 2x f(t) A(t)(2FF_{xx} - 2F_x^2) \\ &\quad + g(t)(6F_{xx}^2 - 8F_x F_{xxx} + 2FF_{xxxx}) = 0, \end{aligned} \tag{4}$$

where $a(t)=\frac{g(t)}{f(t)}$. If we set $A(t)=\frac{e^{-\int l(t)\,\mathrm{d}t}}{2c_1+2\int e^{-\int l(t)\,\mathrm{d}t}f(t)\,\mathrm{d}t}$, (4) can be written as the following variable-coefficient bilinear form

$$\left[D_xD_t+g(t)D_x^4+h(t)xD_x^2+h(t)\frac{\partial}{\partial x}\right](F\cdot F)=0, \tag{5}$$

with the constraint condition

$$g(t)=c_2f(t)e^{-\int l(t)\,\mathrm{d}t}\left[c_1+\int e^{-\int l(t)\,\mathrm{d}t}f(t)\,\mathrm{d}t\right], \tag{6}$$

where $h(t)=\frac{f(t)e^{-\int l(t)\,\mathrm{d}t}}{c_1+\int e^{-\int l(t)\,\mathrm{d}t}f(t)\,\mathrm{d}t}$, c_1 and $c_2\neq 0$ are both arbitrary constants, while constraint (6) is just the condition for (1) to pass the Painlevé test [48], and D_xD_t, D_x^4 and D_x^2 are all bilinear derivative operators [49–51] defined by

$$D_x^mD_t^na\cdot b\equiv\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^m\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^n a(x,t)b(x',t')\bigg|_{x'=x,t'=t}. \tag{7}$$

Therefore, under constraint (6), the bilinear form for (1) is obtained, i.e., (5).

Based on (5), a bilinear auto-BT for (1) with constraint (6) can be presented as

$$\begin{cases}[D_t+g(t)D_x^3+3\lambda(t)g(t)D_x+h(t)xD_x+\delta(t)]F'\cdot F=0,\\ [D_x^2-\lambda(t)]F'\cdot F=0,\end{cases} \tag{8}$$

where $F(x,t)$ and $F'(x,t)$ are two different solutions for (5), and $\lambda(t)$ and $\delta(t)$ are both arbitrary functions of t.

It is noted that the obtained bilinear auto-BT is very useful in deriving the N-solitonic solution in Wronskian form. To illustrate, we can calculate the one-solitonic solution from the seed solution $F'=1$. Without loss of generality, the function $\delta(t)$ in (8) is assumed to zero. Substituting $F'=1$ into (8) yields

$$\begin{cases}F_t+g(t)F_{xxx}+3\lambda(t)g(t)F_x+h(t)xF_x=0,\\ F_{xx}-\lambda(t)F=0,\end{cases} \tag{9}$$

from which the one-solitonic solution of (5) is obtained as

$$F=e^{\theta}+e^{-\theta}, \tag{10}$$

$$\theta=\frac{kx}{c_1+\int e^{-\int l(t)\,\mathrm{d}t}f(t)\,\mathrm{d}t}+\frac{4k^3c_2}{c_1+\int e^{-\int l(t)\,\mathrm{d}t}f(t)\,\mathrm{d}t}, \tag{11}$$

where k is an arbitrary constant. Moreover, according to (10, 11) and $u=w_x$, the one-solitonic solution of (1) in explicit form can be expressed as follows:

$$\begin{aligned}u&=\frac{xe^{-\int l(t)\,\mathrm{d}t}}{c_1+\int e^{-\int l(t)\,\mathrm{d}t}f(t)\,\mathrm{d}t}+12a(t)[\ln F(x,t)]_{xx}\\ &=\frac{xe^{-\int l(t)\,\mathrm{d}t}}{c_1+\int e^{-\int l(t)\,\mathrm{d}t}f(t)\,\mathrm{d}t}+\frac{12k^2c_2e^{-\int l(t)\,\mathrm{d}t}}{c_1+\int e^{-\int l(t)\,\mathrm{d}t}f(t)\,\mathrm{d}t}\mathrm{Sech}^2\left[\frac{k(x+4k^2c_2)}{c_1+\int e^{-\int l(t)\,\mathrm{d}t}f(t)\,\mathrm{d}t}\right].\end{aligned} \tag{12}$$

3 Wronskian Form of the N-Solitonic Solution

Using the Wronskian technique developed by Freeman and Nimmo, we assume that the N-solitonic solution of (1) can be written in the Wronskian form,

$$F^{(N)} = W(\phi_1, \phi_2, \dots, \phi_N) = \begin{vmatrix} \phi_1 & \phi_1^{(1)} & \phi_1^{(2)} & \cdots & \phi_1^{(N-1)} \\ \phi_2 & \phi_2^{(1)} & \phi_2^{(2)} & \cdots & \phi_2^{(N-1)} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \phi_{N-1} & \phi_{N-1}^{(1)} & \phi_{N-1}^{(2)} & \cdots & \phi_{N-1}^{(N-1)} \\ \phi_N & \phi_N^{(1)} & \phi_N^{(2)} & \cdots & \phi_N^{(N-1)} \end{vmatrix}, \tag{13}$$

with

$$\phi_i^{(j)} = \frac{\partial^j \phi_i}{\partial x^j} \quad (j = 1, 2, \dots, N-1), \quad \phi_i = e^{\theta_i} + (-1)^{i+1} e^{-\theta_i},$$

$$\theta_i = \frac{k_i x}{c_1 + \int e^{-\int l(t)\,\mathrm{d}t} f(t)\,\mathrm{d}t} + \frac{4k_i^3 c_2}{c_1 + \int e^{-\int l(t)\,\mathrm{d}t} f(t)\,\mathrm{d}t},$$

and ϕ_i $(i = 1, 2, \dots, N)$ satisfy the partial differential equations

$$\begin{cases} \phi_{ixx} = \lambda_i(t)\phi_i, \\ \phi_{it} = -4g(t)\phi_{ixxx} - h(t)x\phi_{ix}, \end{cases} \tag{14}$$

where $\lambda_i(t) = \frac{k_i^2}{[c_1 + \int e^{-\int l(t)\,\mathrm{d}t} f(t)\,\mathrm{d}t]^2}$ $(i = 1, 2, \dots, N)$ are the functions of t with k_i as arbitrary parameters. If we choose the notation $W(\phi_1, \phi_2, \dots, \phi_N) = (0, 1, 2, \dots, N-1) = (\widehat{N-1})$, then, $F^{(N)} = (\widehat{N-1})$ and the derivatives of $F^{(N)}$ can be easily computed out:

$$F^{(N)} = (\widehat{N-1}), \tag{15}$$

$$F_x^{(N)} = (\widehat{N-2}, N), \tag{16}$$

$$F_{xx}^{(N)} = (\widehat{N-3}, N-1, N) + (\widehat{N-2}, N+1), \tag{17}$$

$$F_{xxx}^{(N)} = (\widehat{N-4}, N-2, N-1, N) + 2(\widehat{N-3}, N-1, N+1) + (\widehat{N-2}, N+2), \tag{18}$$

$$\begin{aligned} F_{xxxx}^{(N)} = {} & (\widehat{N-5}, N-3, N-2, N-1, N) + 3(\widehat{N-4}, N-2, N-1, N+1) \\ & + 2(\widehat{N-3}, N, N+1) + 3(\widehat{N-3}, N-1, N+2) + (\widehat{N-2}, N+3), \end{aligned} \tag{19}$$

and using the second formula of (14), we have

$$\begin{aligned} F_t^{(N)} = {} & -4g(t)\big[(\widehat{N-4}, N-2, N-1, N) - (\widehat{N-3}, N-1, N+1) + (\widehat{N-2}, N+2)\big] \\ & - h(t)x(\widehat{N-2}, N) - \frac{N(N-1)}{2} h(t)(\widehat{N-1}), \end{aligned} \tag{20}$$

$$
\begin{aligned}
F_{xt}^{(N)} = & -4g(t)\big[(\widehat{N-5}, N-3, N-2, N-1, N) - (\widehat{N-3}, N, N+1) \\
& + (\widehat{N-2}, N+3)\big] - h(t)(\widehat{N-2}, N) - h(t)x(\widehat{N-3}, N-1, N) \\
& - h(t)x(\widehat{N-2}, N+1) - \frac{N(N-1)}{2}h(t)(\widehat{N-2}, N).
\end{aligned} \tag{21}
$$

For verifying solution (13) satisfies (5), we substitute (15–21) into (5), yielding

$$
\begin{aligned}
& \left[D_x D_t + g(t)D_x^4 + h(t)xD_x^2 + h(t)\frac{\partial}{\partial x}\right](F^{(N)} \cdot F^{(N)}) \\
& = 2F^{(N)}F_{xt}^{(N)} - 2F_t^{(N)}F_x^{(N)} + g(t)\big[2F^{(N)}F_{xxxx}^{(N)} - 8F_x^{(N)}F_{xxx}^{(N)} + 6(F_{xx}^{(N)})^2\big] \\
& \quad + h(t)x\big[2F^{(N)}F_{xx}^{(N)} - 2(F_x^{(N)})^2\big] + 2h(t)F^{(N)}F_x^{(N)} \\
& = 24g(t)\big[(\widehat{N-1})(\widehat{N-3}, N, N+1) - (\widehat{N-2}, N)(\widehat{N-3}, N-1, N+1) \\
& \quad + (\widehat{N-3}, N-1, N)(\widehat{N-2}, N+1)\big] \\
& = 12g(t)\begin{vmatrix} \widehat{N-3} & \mathbf{0} & N-2 & N-1 & N & N+1 \\ \mathbf{0} & \widehat{N-3} & N-2 & N-1 & N & N+1 \end{vmatrix} = 0,
\end{aligned} \tag{22}
$$

which suggests that solution (13) is indeed an exact analytic solution of (5).

4 An Application of the Wronskian Technique to the Auto-BT

It is known that the auto-BT can give a hierarchy of explicit solutions including the N-solitonic solution. In this section, we will employ the Wronskian technique to verify that the auto-BT (8) between the $(N-1)$- and N-solitonic solutions is indeed satisfied.

Similarly, if we choose $F^{(N)} = (\widehat{N-1})$ and

$$
\begin{aligned}
F^{(N-1)} = W(\phi_1, \phi_2, \ldots, \phi_{N-1}, \tau) = & \begin{vmatrix} \phi_1 & \phi_1^{(1)} & \phi_1^{(2)} & \cdots & \phi_1^{(N-2)} & 0 \\ \phi_2 & \phi_2^{(1)} & \phi_2^{(2)} & \cdots & \phi_2^{(N-2)} & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \phi_{N-1} & \phi_{N-1}^{(1)} & \phi_{N-1}^{(2)} & \cdots & \phi_{N-1}^{(N-2)} & 0 \\ \phi_N & \phi_N^{(1)} & \phi_N^{(2)} & \cdots & \phi_N^{(N-2)} & 1 \end{vmatrix} \\
= & (\widehat{N-2}, \tau),
\end{aligned} \tag{23}
$$

where $\tau = (0, \ldots, 0, 1)^T$, then $F^{(N)}$ denotes the N-solitonic solution and $F^{(N-1)}$ is the $(N-1)$-solitonic solution. Meanwhile, the derivatives of $F^{(N-1)}$ can be written as:

$$
F_x^{(N-1)} = (\widehat{N-3}, N-1, \tau), \tag{24}
$$

$$
F_{xx}^{(N-1)} = (\widehat{N-4}, N-2, N-1, \tau) + (\widehat{N-3}, N, \tau), \tag{25}
$$

$$F_{xxx}^{(N-1)} = (\widehat{N-5}, N-3, N-2, N-1, \tau) + 2(\widehat{N-4}, N-2, N, \tau) + (\widehat{N-3}, N+1, \tau), \tag{26}$$

$$F_{t}^{(N-1)} = -4g(t)\big[(\widehat{N-5}, N-3, N-2, N-1, \tau) - (\widehat{N-4}, N-2, N, \tau) + (\widehat{N-3}, N+1, \tau)\big] - h(t)x(\widehat{N-3}, N-1, \tau) - \frac{(N-1)(N-2)}{2}h(t)(\widehat{N-2}, \tau). \tag{27}$$

Substitution of (15–21) and (23–27) into the auto-BT (8) gives

$$\begin{aligned}
&[D_t + g(t)D_x^3 + 3\lambda(t)g(t)D_x + h(t)xD_x + \delta(t)](F^{(N)}\cdot F^{(N-1)}) \\
&\quad = F_t^{(N)}F^{(N-1)} - F^{(N)}F_t^{(N-1)} + \delta(t)F^{(N)}F^{(N-1)} \\
&\qquad + g(t)\big(F_{xxx}^{(N)}F^{(N-1)} - 3F_{xx}^{(N)}F_x^{(N-1)} + 3F_x^{(N)}F_{xx}^{(N-1)} - F^{(N)}F_{xxx}^{(N-1)}\big) \\
&\qquad + 3\lambda(t)g(t)\left(F_x^{(N)}F^{(N-1)} - F^{(N)}F_x^{(N-1)}\right) + h(t)x\left(F_x^{(N)}F^{(N-1)} - F^{(N)}F_x^{(N-1)}\right) \\
&\quad = 6g(t)\begin{vmatrix} \widehat{N-4} & N-2 & \mathbf{0} & 0 & N-3 & N-1 & N & \tau \\ \mathbf{0} & 0 & \widehat{N-4} & N-2 & N-3 & N-1 & N & 0 \end{vmatrix} \\
&\qquad + 6g(t)\begin{vmatrix} \widehat{N-3} & \mathbf{0} & N-2 & N-1 & N+1 & \tau \\ \mathbf{0} & \widehat{N-3} & N-2 & N-1 & N+1 & 0 \end{vmatrix} = 0,
\end{aligned} \tag{28}$$

and

$$\begin{aligned}
&[D_x^2 - \lambda(t)](F^{(N)}\cdot F^{(N-1)}) \\
&\quad = F_{xx}^{(N)}F^{(N-1)} - 2F_x^{(N)}F_x^{(N-1)} + F^{(N)}F_{xx}^{(N-1)} - \lambda(t)F^{(N)}F^{(N-1)} \\
&\quad = 2(\widehat{N-2}, \tau)(\widehat{N-3}, N-1, N) - 2(\widehat{N-2}, N)(\widehat{N-3}, N-1, \tau) \\
&\qquad + 2(\widehat{N-1})(\widehat{N-3}, N, \tau) \\
&\quad = \begin{vmatrix} \widehat{N-3} & \mathbf{0} & N-2 & N-1 & N & \tau \\ \mathbf{0} & \widehat{N-3} & N-2 & N-1 & N & \tau \end{vmatrix} = 0,
\end{aligned} \tag{29}$$

provided that $\lambda(t) = \frac{K_N^2}{[c_1+\int e^{-\int l(t)\,\mathrm{d}t} f(t)\,\mathrm{d}t]^2}$ and $\delta(t) = (1-N)h(t)$. The above results show that the auto-BT (8) is indeed satisfied.

5 Conclusions

In this paper, we have derived the bilinear form and auto-BT for a perturbed vcKdV equation under a more generalized constraint condition. We have also constructed the N-solitonic solution in the Wronskian form and proved it by direct substitution into the bilinear equation. Further, we have verified that the $(N-1)$- and N-solitonic solutions satisfy the obtained auto-BT. Based on the results above, it has been shown that the Hirota's direct method and the Wronskian technique are also useful in the study of vcKdV models and other variable-coefficient NLEEs.

Acknowledgements We express our sincere thanks to Prof. Y.T. Gao, Ms. J. Li, Ms. L.L. Li, Mr. T. Xu and Mr. H.Q. Zhang for their valuable comments. This work has been supported by the Key Project of Chinese Ministry of Education (No. 106033), by the Specialized Research Fund for the Doctoral Program of Higher Education (No. 20060006024), Chinese Ministry of Education, and by the National Natural Science Foundation of China under Grant No. 60372095.

References

1. Grimshaw, R.H.: Proc. Roy. Soc. A **368**, 359 (1979)
2. Tian, B., Gao, Y.T.: Phys. Plasmas **12**, 054701 (2005)
3. Hong, W.P., Jung, Y.D.: Phys. Lett. A **257**, 149 (1999)
4. Wang, M.L., Wang, Y.M., Zhou, Y.B.: Phys. Lett. A **303**, 45 (2002)
5. Fan, E.G.: Phys. Lett. A **294**, 26 (2002)
6. Sabry, R., Zahranb, M.A., Fan, E.G.: Phys. Lett. A **326**, 93 (2004)
7. Zhang, C.Y., Gao, Y.T., Meng, X.H., Li, J., Xu, T., Wei, G.M., Zhu, H.W.: J. Phys. A **39**, 14353 (2006)
8. Zhu, H.W., Tian, B.: Science paper Online. Chinese Ministry of Education, No. 200701-216, http://www.paper.edu.cn/index.html
9. Yao, Z.Z., Zhang, C.Y., Zhu, H.W., Meng, X.H., Xu, T., Tian, B.: Science paper Online. Chinese Ministry of Education, No. 200701-178, http://www.paper.edu.cn/index.html
10. Cascaval, R.C.: In: Goldstein, G.R., Nagel, R., Romanelli, S. (eds.) Evolution Equations. Lecture Notes on Pure and Applied Mathematics, vol. 234. Dekker, New York (2003)
11. Tian, B., Wei, G.M., Zhang, C.Y., Shan, W.R., Gao, Y.T.: Phys. Lett. A **356**, 8 (2006)
12. Pedley, T.J.: The Fluid Mechanics of Large Blood Vessels. Cambridge University Press, Cambridge (1980)
13. Fung, Y.C.: Biodynamics: Circulation. Springer, New York (1984)
14. Hashizume, Y.: J. Phys. Soc. Jpn. **54**, 3305 (1985)
15. Erbay, H.A., Erbay, S., Dost, S.: Acta Mech. **95**, 87 (1992)
16. Demiray, H.: Math. Comput. Model. **39**, 151 (2004)
17. Bakirtas, I., Demiray, H.: Int. J. Non-Linear Mech. **40**, 785 (2005)
18. Davis, K.B., Mewes, M.O., Andrews, M.R., van Druten, N.J., Durfee, D.S., Kurn, D.M., Ketterle, W.: Phys. Rev. Lett. **75**, 3969 (1995)
19. Bradley, C.C., Sackett, C.A., Tollett, J.J., Hulet, R.G.: Phys. Rev. Lett. **75**, 1687 (1995)
20. Dalfovo, F., Giorgini, S., Pitaevskii, L.P., Stringari, S.: Rev. Mod. Phys. **71**, 463 (1999)
21. Leggett, A.J.: Rev. Mod. Phys. **73**, 307 (2001)
22. Mamun, A.A., Shukla, P.: Phys. Lett. A **290**, 173 (2001)
23. Mamun, A.A., Shukla, P.: Phys. Plasmas **9**, 1468 (2002)
24. Tian, B., Gao, Y.T.: Phys. Lett. A **340**, 243 (2005)
25. Tian, B., Gao, Y.T.: Phys. Lett. A **340**, 449 (2005)
26. Frantzeskakis, D.J., Proukakis, N.P., Kevrekidis, P.G.: Phys. Rev. A **70**, 015601 (2004)
27. Dai, H.H., Huo, Y.: Wave Motion **35**, 55 (2002)
28. Ostrovsky, L., Stepanyants, Y.: Rev. Geophys. **27**, 293 (1989)
29. Holloway, P., Pelinovsky, E., Talipova, T.: J. Geophys. Res. C **104**, 18333 (1999)
30. Grimshaw, R.H., Mitsudera, H.: Stud. Appl. Math. **90**, 75 (1993)
31. Hirota, R.: Direct Methods in Soliton Theory. Springer, Berlin (1980).
32. Freeman, N.C., Nimmo, J.J.: Phys. Lett. A **95**, 1 (1983)
33. Nimmo, J.J.: Phys. Lett. A **99**, 279 (1983)
34. Barnett, M.P., Capitani, J.F., Von Zur Gathen, J., Gerhard, J.: Int. J. Quant. Chem. **100**, 80 (2004)
35. Xie, F.D., Gao, X.S.: Comm. Theor. Phys. **41**, 353 (2004)
36. Tian, B., Gao, Y.T.: Phys. Lett. A **342**, 228 (2005)
37. Tian, B., Gao, Y.T.: Phys. Lett. A **359**, 241 (2006)
38. Tian, B., Gao, Y.T.: Phys. Lett. A **362**, 283 (2007)
39. Gao, Y.T., Tian, B., Zhang, C.Y.: Acta Mech. **182**, 17 (2006)
40. Tian, B., Gao, Y.T., Zhu, H.W.: Phys. Lett. A **366**, 223 (2007)
41. Yan, Z.Y., Zhang, H.Q.: J. Phys. A **34**, 1785 (2001)

42. Hong, W.P.: Phys. Lett. A **361**, 520 (2007)
43. Gao, Y.T., Tian, B.: Phys. Plasmas **13**, 112901 (2006)
44. Gao, Y.T., Tian, B.: Phys. Plasmas (Lett.) **13**, 120703 (2006)
45. Gao, Y.T., Tian, B.: Phys. Lett. A **349**, 314 (2006)
46. Gao, Y.T., Tian, B.: Phys. Lett. A **361**, 523 (2007)
47. Gao, Y.T., Tian, B.: Europhys. Lett. **77**, 15001 (2007)
48. Wei, G.M., Gao, Y.T., Hu, W., Zhang, C.Y.: Eur. Phys. J. B **53**, 343 (2006)
49. Hirota, R.: Phys. Rev. Lett. **27**, 1192 (1971)
50. Hirota, R., Satsumam, J.: J. Phys. Soc. Jpn. **40**, 611 (1978)
51. Hirota, R., Hu, X.B., Tang, X.Y.: J. Math. Anal. Appl. **288**, 326 (2003)